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ENTANGLEMENT AND OTHER QUANTUM CORRELATIONS OF A SINGLE QUDIT STATE AS A RESOURCE FOR QUANTUM TECHNOLOGIES

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The approach to extend the notion of entanglement for characterizing the properties of quantum correlations in the state of a single qudit is presented. New information and entropic inequalities, such as the subadditivity condition, strong subadditivity condition, and monotonicity of relative entropy for a single qudit corresponding to an arbitrary spin state with spin j are discussed. The idea to employ quantum correlations in the single-qudit state, such as the entanglement, for developing a new quantum technique in quantum computing and quantum communication is proposed. Examples of qutrit and qudit with $j = 3/2$ are considered.

Keywords: Quantum computing resource; quantum correlations; subadditivity condition; non-composite systems; single qudit.

1. Introduction

The entanglement phenomenon¹ is associated with quantum correlations in composite systems containing subsystems. Entangled states of the composite systems, e.g., of several qubits are considered as a resource for their employment in quantum technologies like quantum computing, quantum teleportation, and quantum coding².

Quantum correlations of composite systems are also characterized by the discord^{3,4,5}. The notion of discord is associated with information properties of bipartite quantum system, e.g., two qubits. The bipartite system states are determined by the density operator matrix $\rho(1, 2)$, which acts in the Hilbert space H of the system

2 Authors' Names

states, and this Hilbert space has the structure of the tensor product $H = H_1 \otimes H_2$ of the Hilbert spaces of the first and second subsystem states.

Bipartite and multipartite quantum system states have entropic and information characteristics. The von Neumann entropy of the bipartite system state satisfies specific inequalities, which are numerical inequalities for matrix elements of the density matrices of the system and its subsystems.

The aim of our work is to show that the inequalities for the density matrices analogous to the known inequalities^{6,7,8,9,10,11,12,13,14,15} for the composite systems can be obtained also for non-composite systems, e.g., for a single qudit. Also we show that there exist the entanglement properties of the density matrix of the single qudit state, and the notion of mutual information can be extended and applied to the single qudit state. This means that the resource of quantum correlations, which are considered from the viewpoint of the possibility to be applied in quantum technologies, is available not only in composite systems as correlations of their subsystems but in non-composite systems as well.

In this paper, we present some new inequalities following the approach suggested in^{16,17,18,19,20,21}. It is worth mentioning that our approach is coherent with the approach to quantum correlations in the single qudit state discussed in²². Also some aspects of our approach are close to the consideration of Laplace matrices in^{23,24,25}.

This paper is organized as follows.

In section 2, we obtain new entropic inequalities on an example of the qutrit state. We present the inequalities for tomograms of qudit states in section 3 and discuss the monotonicity of the single-qudit-state density matrices in section 4. We find the Bell inequality for the $j = 3/2$ single qudit in section 5 and give our conclusions and perspectives in section 6.

2. Information and entropic inequalities

Before formulating the information and entropic inequalities, first we study linear maps of matrices. Following²⁶, we consider rectangular matrices as vectors. Let matrix a have matrix elements a_{jk} , where $j = 1, 2, \dots, n$ and $k = 1, 2, \dots, m$. We construct the linear map $\hat{V}a = \vec{a}$, where the N -vector \vec{a} has nm components, i.e., $\vec{a} = (a_{11}, a_{12}, \dots, a_{1m}, a_{21}, a_{22}, \dots, a_{2m}, \dots, a_{n1}, a_{n2}, \dots, a_{nm})$.

The inverse map provides a matrix from the N -vector, i.e., $a = \hat{V}^{-1}\vec{a}$.

Using the vectors \vec{a} instead of the matrices a , one can describe the linear maps of the matrices by means of linear transforms of the corresponding vectors \vec{a} . This means that the map of matrix $a \rightarrow b$ denoted as

$$b = \hat{L}a \quad (1)$$

has the matrix form

$$b_J = \sum_{K=1}^N L_{JK} a_K. \quad (2)$$

We consider the case of $m = n$ and introduce some special transforms of the matrix a as follows:

$$b = P_{n-1}aP_{n-1} + P_naP_n = \hat{L}a, \quad (3)$$

where the $n \times n$ -matrix P_n acts on any n -vector $\vec{c} = (c_1, c_2, \dots, c_n)$ as a projector of the form $P_n\vec{c} = (0, 0, \dots, c_n)$. The matrix $P_{n-1} = 1_n - P_n$, where 1_n is the identity $n \times n$ -matrix, and the matrix L_{JK} in this case has some nonzero diagonal matrix elements equal to unity. Non-diagonal matrix elements are equal to zero. The zero diagonal matrix elements read

$$L_{JK} = \delta_{JK}f(J), \quad (4)$$

where the function $f(J)$ is equal to zero for the following values of the arguments

$$f(n) = f(2n) = \dots = f((n-1)n) = f(n^2-n+1) = f(n^2-n+2) = \dots = f(n^2-1) = 0. \quad (5)$$

All the other values of the function $f(J) = 1$.

We denote the matrix L_{JK} (4) with the properties described by equation (5) as a matrix M_2 . For example, if $n = 3$, the 9×9 -matrix M_2 has the block form

$$M_2 = \begin{pmatrix} \mathcal{P}_2 & 0 & 0 \\ 0 & \mathcal{P}_2 & 0 \\ 0 & 0 & \mathcal{P}_1 \end{pmatrix}; \quad \mathcal{P}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \mathcal{P}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (6)$$

If the matrix a is the density matrix, i.e., $a = a^\dagger$, $\text{Tr } a = 1$, and $a \geq 0$, the map given by equation (3) is the positive map. In terms of the matrix M_2 , this map corresponds to the transform of the vector \vec{a} , i.e., $\hat{V}^{-1}(M_2\vec{a})$ is the matrix with the properties of a qutrit density matrix. The $N \times N$ -matrix L_{JK} (4) can be written in the form

$$L_{JK} = (P_{n-1} \otimes P_{n-1} + P_n \otimes P_n)_{JK}. \quad (7)$$

This formula is the partial case to obtain the matrix L_{JK} , which corresponds to the linear transform of the matrix a given in the form

$$a \rightarrow b = \hat{L}a = \sum_s P_s a P_s^\dagger, \quad (8)$$

where P_s are arbitrary matrices. In this case, the matrix L_{JK} reads

$$L_{JK} = \left(\sum_s P_s \otimes P_s^* \right)_{JK}. \quad (9)$$

The index s in (8) and (9) can take values in an arbitrary domain. In the case where the matrices P_s are projectors, formula (9) provides the diagonal matrix L_{JK} .

For matrices a , which are the density matrices, we denote the matrix L_{JK} as M_2 . The transform (9) with projectors P_s yields the positive map (completely positive map) of the matrix a . The positive transform of the density matrix can be interpreted as a decoherence transform, which corresponds to equating some non-diagonal matrix elements of the density matrix to zero.

4 Authors' Names

One can construct another map of the matrix a . This map is given by the transform of the vector $\vec{a} \rightarrow M_1 \vec{a}$, where the matrix M_1 reads

$$M_1 = \begin{pmatrix} \Pi_1 & 0 & 0 \\ 0 & \Pi_2 & 0 \\ 0 & 0 & \Pi_1 \end{pmatrix}; \quad \Pi_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad \Pi_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (10)$$

One can construct another matrix \widetilde{M}_2 of the form

$$\widetilde{M}_2 = M_2 + S_1 - P, \quad (11)$$

where the only nonzero matrix element in $N \times N$ -matrices S_1 and P is

$$(S_1)_{1N} = P_{NN} = 1. \quad (12)$$

Then for the matrix a one has the map

$$a \rightarrow \tilde{a}_2 = \hat{V}^{-1}(\widetilde{M}_2(\hat{V}a)). \quad (13)$$

If a is the density $n \times n$ -matrix, the matrix \tilde{a}_2 is also the density $n \times n$ -matrix, and it has the matrix elements $(\tilde{a}_2)_{jn} = (\tilde{a}_2)_{nj} = 0$ for all $j = 1, 2, \dots, n$.

For an example of the qutrit density matrix a , the matrix \tilde{a}_2 reads

$$\tilde{a}_2 = \begin{pmatrix} a_{11} + a_{33} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (14)$$

In fact, the obtained matrix corresponds to the density 2×2 -matrix of the qubit state. The positive map $a \rightarrow \tilde{a}_2$ provides the map of the qutrit density matrix onto the qubit density matrix $a \rightarrow \tilde{\tilde{a}}$ of the form

$$\tilde{\tilde{a}}_2 = \begin{pmatrix} a_{11} + a_{33} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \quad (15)$$

The matrix M_1 describes another positive map of the qutrit density matrix

$$a \rightarrow a_1 = \hat{V}^{-1}(M_1(\hat{V}a)) = \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & a_{33} \end{pmatrix}. \quad (16)$$

This map can also be modified, if one uses the 9×9 -matrix given in the block form

$$\widetilde{M}_1 = \begin{pmatrix} A & B & 0 \\ 0 & 0 & A \\ 0 & 0 & 0 \end{pmatrix}; \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}; \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (17)$$

Then one has the positive map

$$a \rightarrow \tilde{a}_1 = \hat{V}^{-1}(\widetilde{M}_1(\hat{V}a)) = \begin{pmatrix} a_{11} + a_{22} & a_{13} & 0 \\ a_{31} & a_{33} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (18)$$

This map can be considered as a map of the qutrit density matrix onto the qubit density matrix

$$a \rightarrow \tilde{a}_1 = \begin{pmatrix} a_{11} + a_{22} & a_{13} \\ a_{31} & a_{33} \end{pmatrix}. \quad (19)$$

Following ^{16,27,28,29,30,31} it is easy to check that one has the entropic inequality for the von Neumann entropies corresponding to the subadditivity condition

$$\begin{aligned} & -\text{Tr}\{[\hat{V}^{-1}(M_1(\hat{V}a))] \ln [\hat{V}^{-1}(M_1(\hat{V}a))]\} \\ & -\text{Tr}\{[\hat{V}^{-1}(M_2(\hat{V}a))] \ln [\hat{V}^{-1}(M_2(\hat{V}a))]\} \geq -\text{Tr}\{a \ln a\}. \end{aligned} \quad (20)$$

Using the properties of Tsallis entropy ³² and the definition of deformed logarithm

$$\ln_q(x) = \begin{cases} \frac{x^{q-1}-1}{q-1}, & \text{if } q \neq 1, \\ \ln x, & \text{if } q = 1, \end{cases}$$

we obtain the new inequality by replacing $\ln a \rightarrow \ln_q a$ in (20).

The difference I_q of the left- and right-hand sides of inequality (20) is an analog of the mutual information, which reflects quantum correlations in the qutrit state. Thus, one has the information inequality for the qutrit state

$$\begin{aligned} I_q &= -\text{Tr}\{[\hat{V}^{-1}(M_1(\hat{V}a))] \ln [\hat{V}^{-1}(M_1(\hat{V}a))]\} \\ & -\text{Tr}\{[\hat{V}^{-1}(M_2(\hat{V}a))] \ln [\hat{V}^{-1}(M_2(\hat{V}a))]\} + \text{Tr}\{a \ln a\} \geq 0. \end{aligned} \quad (21)$$

We obtain the deformed mutual information by replacing $\ln a \rightarrow \ln_q a$ in (21).

The form of inequality (21) is preserved for any $N \times N$ -matrix a . The inequality takes place for an arbitrary matrix UaU^\dagger , where U is a unitary matrix. In the case where the unitary matrix U is such a matrix that $UaU^\dagger = a_d$ is a diagonal matrix identified with the probability distribution d_1, d_2, d_3 , the inequality yields the relation valid for the probability distribution of the form

$$\begin{aligned} & -(d_1 + d_2) \ln(d_1 + d_2) - d_3 \ln d_3 - (d_1 + d_3) \ln(d_1 + d_3) - d_2 \ln d_2 \\ & \geq -d_1 \ln d_1 - d_2 \ln d_2 - d_3 \ln d_3, \end{aligned} \quad (22)$$

or

$$-(d_1 + d_2) \ln(d_1 + d_2) - (d_1 + d_3) \ln(d_1 + d_3) \ln(d_1 + d_3) \geq -d_1 \ln d_1. \quad (23)$$

It is easy to see that if we consider the $n \times n$ -matrix a for an arbitrary $n \geq 4$, the map discussed provides the $(n-1) \times (n-1)$ -matrix \tilde{a}_2 and 2×2 -matrix \tilde{a}_1

$$\tilde{a}_2 = \begin{pmatrix} a_{11} + a_{nn} & a_{12} & \dots & a_{1n-1} \\ a_{21} & a_{22} & \dots & a_{2n-1} \\ \dots & \dots & \dots & \dots \\ a_{n-11} + a_{n-1,2} & a_{12} & \dots & a_{n-1n-1} \end{pmatrix}, \quad (24)$$

$$\tilde{a}_1 = \begin{pmatrix} a_{11} + a_{22} + \dots + a_{n-1n-1} & a_{1n} \\ a_{n1} & a_{nn} \end{pmatrix}. \quad (25)$$

If the matrix a is the density matrix of a qudit state with $j = (n-1)/2$, the matrix \tilde{a}_2 is an analog of the density matrix of the qudit state with $j = (n-2)/2$, and the matrix \tilde{a}_1 is an analog of the density matrix of the qubit state.

In this case, the subadditivity condition extending inequality (20) reads

$$-\text{Tr}(\tilde{a}_1 \ln \tilde{a}_1) - \text{Tr}(\tilde{a}_2 \ln \tilde{a}_2) \geq -\text{Tr}(a \ln a). \quad (26)$$

Deformed inequality is obtained by replacing $\ln a$ with $\ln_q a$. We should point out that inequality (26) takes place for a single qudit, and the Hilbert space H of the qudit state is not considered as the tensor-product of the Hilbert spaces of two physical subsystems. The corresponding matrices of the map \tilde{M}_1 and \tilde{M}_2 can be also given in an explicit form.

It is worth pointing out that the other different entropic inequalities can be obtained for arbitrary density matrices of composite and non-composite systems if one uses projectors of different ranks in (3).

3. No-signaling property of the single-qudit-state tomogram

For a composite system, e.g., for a two-qudit state with the density $N \times N$ -matrix $\rho(1, 2)$, the joint tomographic probability distribution $w(m_1, m_2, u)$ is defined as

$$w(m_1, m_2, u) = \langle m_1 m_2 | u \rho(1, 2) u^\dagger | m_1 m_2 \rangle. \quad (27)$$

The probability distribution, where $m_1 = -j_1, -j_1 + 1, \dots, j_1$, $m_2 = -j_2, -j_2 + 1, \dots, j_2$, and u is a unitary $N \times N$ -matrix with $N = (2j_1 + 1)(2j_2 + 1)$ is the joint probability distribution of two random spin projections m_1 and m_2 . The probability distribution depends on the unitary matrix u .

This probability distribution has the property of no-signaling; this means that for $u = u_1 \otimes u_2$, where u_1 and u_2 are unitary local transforms in Hilbert spaces of the first and second qudits, respectively, the marginal probability distributions

$$w_1(m_1, u) = \sum_{m_2=-j_2}^{j_2} w(m_1, m_2, u), \quad (28)$$

$$w_2(m_2, u) = \sum_{m_1=-j_1}^{j_1} w(m_1, m_2, u) \quad (29)$$

depend on their own local unitary transforms only; this means that

$$w_1(m_1, u = u_1 \otimes u_2) \equiv w_1(m_1, u_1), \quad (30)$$

$$w_2(m_2, u = u_1 \otimes u_2) \equiv w_2(m_2, u_2). \quad (31)$$

Since an arbitrary Hermitian nonnegative $N \times N$ -matrix $A = A^\dagger$, such that $\text{Tr} A = 1$, can be considered as the density matrix of a single qudit, one can formulate an analogous no-signaling property of such a matrix though it does not describe

the state of a composite system. Thus, one has the property

$$\frac{\partial}{\partial u_2} \sum_{m_2=-j_2}^{j_2} \langle m_1 m_2 | u A u^\dagger | m_1 m_2 \rangle_{u=u_1 \times u_2} = 0, \quad (32)$$

$$\frac{\partial}{\partial u_1} \sum_{m_1=-j_1}^{j_1} \langle m_1 m_2 | u A u^\dagger | m_1 m_2 \rangle_{u=u_1 \times u_2} = 0, \quad (33)$$

If $N \neq (2j_1+1)(2j_2+1)$, one can replace the matrix A in equalities (32) and (33) with the $\tilde{N} \times \tilde{N}$ -matrix $\tilde{A} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$, where $\tilde{N} = (2j_1+1)(2j_2+1)$. The matrix \tilde{A} is Hermitian, nonnegative, and $\text{Tr } \tilde{A} = \text{Tr } A = 1$. Then the property of no-signaling given by equalities (32) and (33) is valid for the matrix \tilde{A} .

As an example, we consider the 5×5 -matrix ρ , which is identified with the density matrix of a qudit with $j = 2$, i.e., $\rho_{mm'}$, where m and m' are spin projections $-2, -1, 0, 1, 2$. We construct the 6×6 -matrix $\tilde{\rho} = \begin{pmatrix} \rho & 0 \\ 0 & 0 \end{pmatrix}$ and consider this matrix as the density matrix of a qubit–qutrit system. This means that we use the map for the basis in the six-dimensional Hilbert space as

$$\begin{aligned} | -2 \rangle &\equiv | 1/2, 1 \rangle, \quad | -1 \rangle \equiv | 1/2, 0 \rangle, \quad | 0 \rangle \equiv | 1/2, 1 \rangle, \\ | 1 \rangle &\equiv | -1/2, 1 \rangle, \quad | 2 \rangle \equiv | -1/2, 0 \rangle, \quad | an \rangle \equiv | -1/2, -1 \rangle, \end{aligned}$$

where the vector $| an \rangle$ is an additional basis vector in the six-dimensional Hilbert space, corresponding to zero matrix elements in the matrix $\tilde{\rho}$. Then we can write the no-signaling property in the form

$$\frac{\partial}{\partial u(2)} \sum_{m_2=-1}^1 \langle m_1 m_2 | u(2) \times u(3) \tilde{\rho} u^\dagger(2) \times u^\dagger(3) | m_1 m_2 \rangle = 0, \quad (34)$$

$$\frac{\partial}{\partial u(3)} \sum_{m_1=-1/2}^{1/2} \langle m_1 m_2 | u(2) \times u(3) \tilde{\rho} u^\dagger(2) \times u^\dagger(3) | m_1 m_2 \rangle = 0, \quad (35)$$

where $u(2)$ is the unitary 2×2 -matrix and $u(3)$ is the unitary 3×3 -matrix.

In fact, the equalities obtained are new properties of arbitrary Hermitian non-negative matrices with unit trace. Analogous no-signaling properties can be obtained with respect to an arbitrary product decomposition of unitary $N \times N$ -matrix $u = u_1 \times u_2 \times \cdots \times u_m$, if $N = n_1 n_2 \cdots n_m$.

4. Monotonicity property of density matrices of a single qudit and two qubits

It is known that the density matrices of bipartite quantum systems $\rho(1, 2)$ and $\sigma(1, 2)$ have the monotonicity property², i.e., their relative von Neumann entropies satisfy the inequality

$$\text{Tr } \rho(1, 2) (\ln \rho(1, 2) - \ln \sigma(1, 2)) \geq \text{Tr } \rho(2) (\ln \rho(2) - \ln \sigma(2)), \quad (36)$$

where $\rho(2) = \text{Tr}_1 \rho(1, 2)$ and $\sigma(2) = \text{Tr}_1 \sigma(1, 2)$. The density matrices are the matrices of the density operators $\hat{\rho}(1, 2)$ and $\hat{\sigma}(1, 2)$ acting in the Hilbert space $H = H_1 \times H_2$. The spaces H_1 and H_2 are Hilbert spaces of the subsystem states of the bipartite quantum system. In fact, the monotonicity property (36) can be obtained for an arbitrary $N \times N$ -matrix R with the properties $R^\dagger = R$, $\text{Tr } R = 1$, and $R \geq 0$.

The matrix R can be considered, e.g., as the density matrix of a single qudit state, such that $N = 2j + 1$; this means that the density matrix R describes the state of the system, which has no subsystems.

To obtain the inequality called the monotonicity of the single qudit state, we apply the portrait method of ^{33,34}. We map the basic vectors of the Hilbert space $H \mid -j \rangle \mid -j + 1 \rangle \cdots \mid j = 1 \rangle \mid j \rangle$ onto vectors $\mid m_1 m_2 \rangle$, where $m_1 = j_1, -j_1 + 1, \dots, j_1 - 1, j_1$ and $m_2 = j_2, -j_2 + 1, \dots, j_2 - 1, j_2$. Then, in view of the identity of the density matrix properties with respect to the numerical relations containing the matrix elements of the matrices, which do not depend on the tensor-product structure of the Hilbert spaces of states for bipartite qudit system or single qudit system, we obtain the inequality under consideration. It was proved ¹² that equality in (36) takes place if and only if the equality

$$\ln \rho(1, 2) - \ln \sigma(1, 2) = \ln \rho(2) - \ln \sigma(2) \quad (37)$$

holds.

Now we derive an analog of this condition for the monotonicity property for a single qudit state. We formulate the main result on the example of the qudit ($j = 3/2$) state. The density matrices $\rho^{(3/2)}$ and $\sigma^{(3/2)}$ of the qudit states have the form

$$\rho^{(3/2)} = \begin{pmatrix} \rho_{3/2 \ 3/2} & \rho_{3/2 \ 1/2} & \rho_{3/2 \ -1/2} & \rho_{3/2 \ -3/2} \\ \rho_{1/2 \ 3/2} & \rho_{1/2 \ 1/2} & \rho_{1/2 \ -1/2} & \rho_{1/2 \ -3/2} \\ \rho_{-1/2 \ 3/2} & \rho_{-1/2 \ 1/2} & \rho_{-1/2 \ -1/2} & \rho_{-1/2 \ -3/2} \\ \rho_{-3/2 \ 3/2} & \rho_{-3/2 \ 1/2} & \rho_{-3/2 \ -1/2} & \rho_{-3/2 \ -3/2} \end{pmatrix}, \quad (38)$$

$$\sigma^{(3/2)} = \begin{pmatrix} \sigma_{3/2 \ 3/2} & \sigma_{3/2 \ 1/2} & \sigma_{3/2 \ -1/2} & \sigma_{3/2 \ -3/2} \\ \sigma_{1/2 \ 3/2} & \sigma_{1/2 \ 1/2} & \sigma_{1/2 \ -1/2} & \sigma_{1/2 \ -3/2} \\ \sigma_{-1/2 \ 3/2} & \sigma_{-1/2 \ 1/2} & \sigma_{-1/2 \ -1/2} & \sigma_{-1/2 \ -3/2} \\ \sigma_{-3/2 \ 3/2} & \sigma_{-3/2 \ 1/2} & \sigma_{-3/2 \ -1/2} & \sigma_{-3/2 \ -3/2} \end{pmatrix}, \quad (39)$$

Then we can write the monotonicity property explicitly

$$\begin{aligned} & \text{Tr} \left[\rho^{(3/2)} \left(\ln \rho^{(3/2)} - \ln \sigma^{(3/2)} \right) \right] \\ & \geq \text{Tr} \left\{ \begin{pmatrix} \rho_{3/2 \ 3/2} + \rho_{-1/2 \ -1/2} & \rho_{3/2 \ 1/2} + \rho_{1/2 \ -3/2} \\ \rho_{1/2 \ 3/2} + \rho_{-3/2 \ -1/2} & \rho_{1/2 \ 1/2} + \rho_{-3/2 \ -3/2} \end{pmatrix} \right. \\ & \times \left[\ln \begin{pmatrix} \rho_{3/2 \ 3/2} + \rho_{-1/2 \ -1/2} & \rho_{3/2 \ 1/2} + \rho_{1/2 \ -3/2} \\ \rho_{1/2 \ 3/2} + \rho_{-3/2 \ -1/2} & \rho_{1/2 \ 1/2} + \rho_{-3/2 \ -3/2} \end{pmatrix} \right. \\ & \left. \left. - \ln \begin{pmatrix} \sigma_{3/2 \ 3/2} + \sigma_{-1/2 \ -1/2} & \sigma_{3/2 \ 1/2} + \sigma_{1/2 \ -3/2} \\ \sigma_{1/2 \ 3/2} + \sigma_{-3/2 \ -1/2} & \sigma_{1/2 \ 1/2} + \sigma_{-3/2 \ -3/2} \end{pmatrix} \right] \right\}. \quad (40) \end{aligned}$$

Also the inequality holds if one makes all possible permutations of the indices $3/2, 1/2, -1/2, -3/2$ of the matrix elements of matrices $\rho^{(3/2)}$ and $\sigma^{(3/2)}$.

As the other example, we consider bipartite system of two qubits with the density matrix $\rho(1, 2)$

$$\rho(1, 2) = \begin{pmatrix} \rho_{1/2 \ 1/2 \ 1/2 \ 1/2} & \rho_{1/2 \ 1/2 \ 1/2 \ -1/2} & \rho_{1/2 \ 1/2 \ -1/2 \ 1/2} & \rho_{1/2 \ 1/2 \ -1/2 \ -1/2} \\ \rho_{1/2 \ -1/2 \ 1/2 \ 1/2} & \rho_{1/2 \ -1/2 \ 1/2 \ -1/2} & \rho_{1/2 \ -1/2 \ -1/2 \ 1/2} & \rho_{1/2 \ -1/2 \ -1/2 \ -1/2} \\ \rho_{-1/2 \ 1/2 \ 1/2 \ 1/2} & \rho_{-1/2 \ 1/2 \ 1/2 \ -1/2} & \rho_{-1/2 \ 1/2 \ -1/2 \ 1/2} & \rho_{-1/2 \ 1/2 \ -1/2 \ -1/2} \\ \rho_{-1/2 \ -1/2 \ 1/2 \ 1/2} & \rho_{-1/2 \ -1/2 \ 1/2 \ -1/2} & \rho_{-1/2 \ -1/2 \ -1/2 \ 1/2} & \rho_{-1/2 \ -1/2 \ -1/2 \ -1/2} \end{pmatrix} \quad (41)$$

and the density matrix $\sigma(1, 2)$ of the same form

$$\sigma(1, 2) = \begin{pmatrix} \sigma_{1/2 \ 1/2 \ 1/2 \ 1/2} & \sigma_{1/2 \ 1/2 \ 1/2 \ -1/2} & \sigma_{1/2 \ 1/2 \ -1/2 \ 1/2} & \sigma_{1/2 \ 1/2 \ -1/2 \ -1/2} \\ \sigma_{1/2 \ -1/2 \ 1/2 \ 1/2} & \sigma_{1/2 \ -1/2 \ 1/2 \ -1/2} & \sigma_{1/2 \ -1/2 \ -1/2 \ 1/2} & \sigma_{1/2 \ -1/2 \ -1/2 \ -1/2} \\ \sigma_{-1/2 \ 1/2 \ 1/2 \ 1/2} & \sigma_{-1/2 \ 1/2 \ 1/2 \ -1/2} & \sigma_{-1/2 \ 1/2 \ -1/2 \ 1/2} & \sigma_{-1/2 \ 1/2 \ -1/2 \ -1/2} \\ \sigma_{-1/2 \ -1/2 \ 1/2 \ 1/2} & \sigma_{-1/2 \ -1/2 \ 1/2 \ -1/2} & \sigma_{-1/2 \ -1/2 \ -1/2 \ 1/2} & \sigma_{-1/2 \ -1/2 \ -1/2 \ -1/2} \end{pmatrix}. \quad (42)$$

The standard monotonicity property is given by the inequality

$$\text{Tr}\{\rho(1, 2)[\ln \rho(1, 2) - \ln \sigma(1, 2)]\} \geq \text{Tr}\{\rho(2)[\ln \rho(2) - \ln \sigma(2)]\}, \quad (43)$$

where

$$\text{Tr}_1 \rho(1, 2) = \rho(2) = \begin{pmatrix} \rho_{1/2 \ 1/2 \ 1/2 \ 1/2} + \rho_{-1/2 \ 1/2 \ -1/2 \ 1/2} & \rho_{1/2 \ 1/2 \ 1/2 \ -1/2} + \rho_{-1/2 \ 1/2 \ -1/2 \ -1/2} \\ \rho_{1/2 \ -1/2 \ 1/2 \ 1/2} + \rho_{-1/2 \ -1/2 \ -1/2 \ 1/2} & \rho_{1/2 \ -1/2 \ 1/2 \ -1/2} + \rho_{-1/2 \ -1/2 \ -1/2 \ -1/2} \end{pmatrix} \quad (44)$$

$$\text{Tr}_1 \sigma(1, 2) = \sigma(2) = \begin{pmatrix} \sigma_{1/2 \ 1/2 \ 1/2 \ 1/2} + \sigma_{-1/2 \ 1/2 \ -1/2 \ 1/2} & \sigma_{1/2 \ 1/2 \ 1/2 \ -1/2} + \sigma_{-1/2 \ 1/2 \ -1/2 \ -1/2} \\ \sigma_{1/2 \ -1/2 \ 1/2 \ 1/2} + \sigma_{-1/2 \ -1/2 \ -1/2 \ 1/2} & \sigma_{1/2 \ -1/2 \ 1/2 \ -1/2} + \sigma_{-1/2 \ -1/2 \ -1/2 \ -1/2} \end{pmatrix}. \quad (45)$$

But there exists the other monotonicity inequality of the form

$$\text{Tr}\{\rho(1, 2)[\ln \rho(1, 2) - \ln \sigma(1, 2)]\} \geq \text{Tr}\{\rho'(2)[\ln \rho'(2) - \ln \sigma'(2)]\}, \quad (46)$$

where the 2×2 -matrix $\rho'(2)$ is not obtained by the partial trace over degrees of freedom of the first subsystem of the matrix $\rho(1, 2)$. For example, consider the

matrices $\rho'(2)$ and $\sigma'(2)$

$$\rho'(2) = \begin{pmatrix} \rho_{1/2 \ -1/2 \ 1/2 \ -1/2} + \rho_{1/2 \ 1/2 \ 1/2 \ 1/2} & \rho_{1/2 \ -1/2 \ -1/2 \ 1/2} + \rho_{1/2 \ 1/2 \ -1/2 \ -1/2} \\ \rho_{-1/2 \ 1/2 \ 1/2 \ -1/2} + \rho_{-1/2 \ -1/2 \ 1/2 \ 1/2} & \rho_{-1/2 \ 1/2 \ -1/2 \ 1/2} + \rho_{-1/2 \ -1/2 \ -1/2 \ -1/2} \end{pmatrix}, \quad (47)$$

$$\sigma'(2) = \begin{pmatrix} \sigma_{1/2 \ -1/2 \ 1/2 \ -1/2} + \sigma_{1/2 \ 1/2 \ 1/2 \ 1/2} & \sigma_{1/2 \ -1/2 \ -1/2 \ 1/2} + \sigma_{1/2 \ 1/2 \ -1/2 \ -1/2} \\ \sigma_{-1/2 \ 1/2 \ 1/2 \ -1/2} + \sigma_{-1/2 \ -1/2 \ 1/2 \ 1/2} & \sigma_{-1/2 \ 1/2 \ -1/2 \ 1/2} + \sigma_{-1/2 \ -1/2 \ -1/2 \ -1/2} \end{pmatrix}. \quad (48)$$

The monotonicity property (46) with matrices $\rho'(2)$ (47) and $\sigma'(2)$ (48) reflects quantum correlations in the system of two qubits. The involved 2×2 -matrices $\rho'(2)$ and $\sigma'(2)$ are not the density matrices of the first and second qubits. These matrices contain contributions of the density matrices of the both subsystems.

One can present other matrices analogous but different from matrices $\rho'(2)$ and $\sigma'(2)$ and different from the standard density matrices $\rho(2)$ and $\sigma(2)$, which satisfy the monotonicity inequality. If due to the property (43), the inequality converts to the equality, all the matrices $\rho(2)$, $\sigma(2)$, $\rho'(2)$, $\sigma'(2)$, etc. satisfy equality (37)

$$[\ln \rho(2) - \ln \sigma(2)] = [\ln \rho'(2) - \ln \sigma'(2)]. \quad (49)$$

5. Bell inequalities and their violation for states of a qudit with $j = 3/2$

The developed approach provides the possibility to extend the discussion of Bell inequalities, e.g., considered in ³⁵ for two qubits to the case of a qudit with $j = 3/2$ (as it was mentioned in ³³).

Since purely numerical properties of matrices (which are tables containing real or complex numbers) do not depend on the structure of Hilbert spaces and operators acting in these Hilbert spaces having the matrix representations under consideration, one can obtain relations for the functions of the matrix elements, which are universal and valid for many systems represented by the numerical matrices.

We demonstrate the known example of such a numerical 4×4 -matrix

$$\rho = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}. \quad (50)$$

If this matrix is identified with the density matrix of the state with density operator $\hat{\rho}(1,2)$ of two qubits, where the basis in the Hilbert space $H = H_1 \times H_2$ has the product form $|m_1 m_2\rangle = |m_1\rangle |m_2\rangle$ with spin projection $m_{1,2} = \pm 1/2$, the matrix ρ (50) corresponds to the pure state

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|1/2 \ 1/2\rangle + |-1/2 \ -1/2\rangle). \quad (51)$$

On the other hand, one can consider the matrix ρ as the density matrix of the state $|\varphi\rangle$ of a single qudit with $j = 3/2$. In this case, the state $|\varphi\rangle$ is the vector in the Hilbert space $H_{3/2}$, and there is no decomposition of this space into “natural” product of two Hilbert spaces.

We use the basis vectors $|m\rangle$, where $m = -3/2, -1/2, 1/2, 3/2$, to construct the matrix ρ (50) containing the matrix elements of the density operator $\hat{\rho}$ as $\rho_{mm'} = \langle m | \hat{\rho} | m' \rangle$. The matrix ρ (50) corresponds to the pure state $|\varphi\rangle$ of a single qudit with $j = 3/2$, which is a linear superposition of two states

$$|\varphi\rangle = \frac{1}{\sqrt{2}} (|3/2\rangle + |-3/2\rangle). \quad (52)$$

From the viewpoint of reflecting the physical properties of the systems, the operators $\hat{\rho}(1, 2)$ and $\hat{\rho}$ and the Hilbert spaces $H = H_1 \times H_2$ and $H_{3/2}$ are quite different, but the numerical matrix ρ given by (50) is universal and the same for the two different physical situations. In view of this observation, we discuss the properties which depend only on the numerical properties of the matrix ρ .

The entanglement phenomenon can be associated with the property of the matrix ρ (50) represented in a form of the convex sum

$$\rho = \sum_k p_k \rho_1^{(k)} \otimes \rho_2^{(k)}, \quad p_k \geq 0, \quad \sum p_k = 1, \quad (53)$$

where Hermitian nonnegative matrices $\rho_1^{(k)}$ and $\rho_2^{(k)}$ are 2×2 -matrices and $\text{Tr} \rho_1^{(k)} = \text{Tr} \rho_2^{(k)} = 1$. The Peres–Horodecki *ppt*-criterion can be formulated for the matrix ρ without using the local structure of the states in Hilbert spaces H or $H_{3/2}$. It is obvious that the necessary condition for the existence of the decomposition of numerical matrix ρ (53) written in a generic form

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{pmatrix}, \quad (54)$$

where the matrix elements ρ_{jk} ($j, k = 1, 2, 3, 4$) are complex numbers, is given by the following statement. If the numerical matrix ρ (54) can be presented in

the form (53), one has the consequence that the matrix $\tilde{\rho} = \begin{pmatrix} \rho_{11} & \rho_{21} & \rho_{13} & \rho_{23} \\ \rho_{12} & \rho_{22} & \rho_{14} & \rho_{24} \\ \rho_{31} & \rho_{41} & \rho_{33} & \rho_{43} \\ \rho_{32} & \rho_{42} & \rho_{34} & \rho_{44} \end{pmatrix}$

is nonnegative Hermitian matrix with $\text{Tr} \tilde{\rho} = 1$. If the matrix ρ is considered as the density matrix of two qubits, i.e., is identified with numerical matrix $\langle m_1 m_2 | \hat{\rho}(1, 2) | m'_1 m'_2 \rangle$, the formulated statement is the *ppt*-criterion.

The statement formulated is valid also in the case where the matrix ρ is identified with the density matrix of the qudit state with $j = 3/2$.

For the matrix ρ (54), one can formally introduce a unitary tomographic probability distribution

$$w_n(u) = \langle n | u \rho u^\dagger | n \rangle, \quad n = 1, 2, 3, 4, \quad (55)$$

where u is the unitary 4×4 -matrix.

In the case where ρ is identified with the density matrix $\rho_{m_1 m_2 m'_1 m'_2}$, the distribution $w_{m_1 m_2}(u)$ is the unitary tomogram of the two-qubit state.

In the case where the matrix ρ is identified with the density matrix of a single qudit state with $j = 3/2$, the distribution (55) is identified with the unitary tomogram $w_n(u)$ of the spin-3/2 state.

Numerically both distributions $w_m(u)$ and $w_{m_1 m_2}(u)$ coincide, but the tomograms have different physical interpretations.

For the case $u = u_1 \times u_2$, where u_1 and u_2 are the spin-1/2 representation matrices depending on the Euler angles (φ, θ) , which provide the unit vector $\vec{n} = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$, the tomogram $w_{m_1 m_2}(\vec{n}_1, \vec{n}_2)$ is the joint probability distribution of two random spin-1/2 projections m_1 and m_2 onto the corresponding quantization directions \vec{n}_1 and \vec{n}_2 .

In the case of ρ identified with the $j = 3/2$ -qudit state, the tomogram $w_m(u)$, where u is the matrix of unitary irreducible representation, converts to the tomogram $w_m(\vec{n})$, which is the probability to have one random spin-3/2 projection m onto the quantization axis \vec{n} .

We point out that both tomograms $w_m(u)$ and $w_{m_1 m_2}(u)$ are the particular values of the distribution $w_n(u)$, where one chooses the particular subgroups of the unitary $u(4)$ group and uses various maps of matrix element indices.

For the two-qubit case, one uses the bijective map

$$1/2 \ 1/2 \leftrightarrow 1, \quad 1/2 \ -1/2 \leftrightarrow 2, \quad -1/2 \ 1/2 \leftrightarrow 3, \quad -1/2 \ -1/2 \leftrightarrow 4.$$

For the $j = 3/2$ -qudit case, one uses the bijective map

$$3/2 \leftrightarrow 1, \quad 1/2 \leftrightarrow 2, \quad -1/2 \leftrightarrow 3, \quad -3/2 \leftrightarrow 4.$$

Now we formulate an extension of the Bell inequality (in fact, CHSH inequality) known for the two-qubit case in the tomographic form to be used to apply the approach to obtain the inequality for an arbitrary matrix ρ (54). Also we obtain the inequality for the $j = 3/2$ -qudit state.

The general approach uses the tomogram $w_n(u)$.

We construct the stochastic 4×4 -matrix M for which four columns are the probability vectors $\vec{w}(u_k)$, $k = 1, 2, 3, 4$ with four components $w_n(u)$. If the matrix ρ has the structure (53), the probability vector $\vec{w}(u)$ is also of a special form. To demonstrate the inequality, first we consider 4×4 -matrix M of the form of the tensor product of two stochastic 2×2 -matrices

$$M = \begin{pmatrix} x & y \\ 1-x & 1-y \end{pmatrix} \otimes \begin{pmatrix} z & t \\ 1-z & 1-t \end{pmatrix}, \quad 0 \leq x, y, z, t \leq 1. \quad (56)$$

Then, it can be checked that the function $B(x, y, z, t) = \text{Tr}(IM)$, where $I = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix}$ satisfies the Laplace equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial t^2} \right) B(x, y, z, t) = 0. \quad (57)$$

This means that the function $B(x, y, z, t)$ has the maximum and minimum values on the boundary of the cube in the four-dimensional space. Thus, taking values of the function $B(x, y, z, t)$ for arguments zero or one, we obtain the inequality

$$|B(x, y, z, t)| \leq 2, \quad (58)$$

which is easy to check. Using the inequality for the convex sum $|\sum_k p_k z_k| \leq \sum_k |z_k|$, we obtain the inequality for the matrix $M = \sum_k p_k M_k$, where M_k is given by (56) with x, y, z, t replaced by x_k, y_k, z_k, t_k and $0 \leq p_k \leq 1$. The inequality for this matrix reads

$$|\text{Tr}(IM)| \leq 2. \quad (59)$$

Now we consider the 4×4 -matrix using four vectors $\vec{w}(u_k)$, i.e.,

$$M(u_1 u_2 u_3 u_4) = \begin{pmatrix} w_1(u_1) & w_1(u_2) & w_1(u_3) & w_1(u_4) \\ w_2(u_1) & w_2(u_2) & w_2(u_3) & w_2(u_4) \\ w_3(u_1) & w_3(u_2) & w_3(u_3) & w_3(u_4) \\ w_4(u_1) & w_4(u_2) & w_4(u_3) & w_4(u_4) \end{pmatrix}. \quad (60)$$

The matrix M is the stochastic matrix. We choose specific matrices u_k of the product form

$$u_1 = u_a \otimes u_b, \quad u_2 = u_a \otimes u_c, \quad u_3 = u_d \otimes u_a, \quad u_4 = u_d \otimes u_c, \quad (61)$$

where the matrices u_a, u_b, u_c , and u_d are arbitrary unitary 2×2 -matrices. In this particular case, one can construct the number

$$B(u_a, u_b, u_c, u_d) = \text{Tr}(IM(u_1, u_2, u_3, u_4)). \quad (62)$$

If the 4×4 -matrix ρ has the structure (53), the number $B(u_a, u_b, u_c, u_d)$ (62) satisfies the inequality

$$|B(u_a, u_b, u_c, u_d)| \leq 2. \quad (63)$$

It is an analog of the CHSH³⁵ inequality proved in view of generic inequality (58). Inequality (63) is valid independently of any product structure of the Hilbert space.

If the matrix ρ can be presented in the form (53), one has an extended Bell inequality (63) for arbitrary unitary 2×2 -matrices u_a, u_b, u_c , and u_d . From this observation follows the statement that for the density matrix of qudit with $j = 3/2$ of the form (53) one has the Bell inequality. We call a such density matrix of the qudit state the separable matrix. If the inequality is violated, it is easy to show that

one has the Cerelson bound $2\sqrt{2}$ ³⁶ for the number $B(u_a, u_b, u_c, u_d)$ calculated for a single qudit state with $j = 3/2$.

The physical interpretation of the Bell inequality and its violation introduced for $j = 3/2$ is different from the standard case of two qubits. For two qubits, quantum correlations exist for the degrees of freedom of two different subsystems of qubits. For a qudit with $j = 3/2$, quantum correlations exist for degrees of freedom (spin projections) of the same system. The violation of Bell inequalities for two qubits was checked experimentally³⁷, and one can try to check the violation of Bell inequalities for qudit with $j = 3/2$.

6. Conclusions

To conclude, we list the main results of our work.

We showed that quantum correlations of the qudit systems expressed in terms of the inequalities for the system density matrices, such as the subadditivity condition and Bell inequalities (known for composite systems), exist also for non-composite systems (like a single qudit). This means that the inequalities for densities matrices, in fact, are universal inequalities for arbitrary nonnegative Hermitian matrices with unit trace.

The quantum correlations of composite quantum systems like systems of N qubits are known to provide the resource for quantum technologies, e.g., for quantum computing. Since analogous quantum correlations exist in the non-composite systems like single qudits, there is a possibility to use them as a resource for quantum technologies as well. For example, the density matrix of the N qubit system and the density matrix of qudit with $j = (N-1)/2$ contain an equivalent resource of the quantum correlations. The problem of finding the possibility to use this resource needs an extra clarification.

It is worth pointing out that analogs of particular matrix inequalities, obtained here, can be extended using other projectors in formula (3) for positive maps. Also the deformed matrix inequalities expressed in terms of q -logarithm can be obtained for density $N \times N$ -matrices of arbitrary composite and non-composite quantum systems of qudits, and this will be done in the future publication.

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16 *Authors' Names*

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